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Compact solitary waves in linearly elastic chains with non-smooth on-site potential*

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Abstract

It was recently observed by Saccomandi and Sgura that one-dimensional chains with nonlinear elastic interaction and regular on-site potential can support compact solitary waves, i.e. travelling solitary waves with strictly compact support. In this paper, we show that the same applies to chains with linear elastic interaction and an on-site potential which is continuous but non-smooth at minima. Some different features arise; in particular, the speed of compact solitary waves is not uniquely fixed by the equation. We also discuss several generalizations of our findings.

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1. Introduction

Solitons—both dynamical and topological—are among the main concepts organizing our understanding of nonlinear phenomena [5, 7, 11, 14, 19, 24, 37].

Solitons are highly local objects, i.e. they differ substantially from a trivial solution (a vacuum in field theoretic language) only in a small region of space. They are however smooth solutions, and as such the approach to trivial states can only be asymptotic, albeit very fast (exponential).

It was shown by Rosenau and Hyman [29] that completely localized (non-smooth) solitons can also exist; they have compact support—i.e. they differ from trivial solutions only in a compact region of space—and were hence termed *compactons*.

In the same way as one can have solitary waves which are not necessarily solitons in a proper mathematical sense (see, e.g., [5] for a discussion), but in many cases are physically equally interesting, one can have *compact solitary waves* which are not necessarily compactons

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in the proper mathematical sense, but are equally interesting physically (a large part of physical literature on compactons deals, in strict mathematical sense, with compact solitary waves). We also note that in the same way as one can have *topological solitons* [7, 11] and topological solitary waves, one can have *topological compactons* (that is, topological solitons with compact support; for theories with degenerate vacua, the support is the region in which the field is not in any of them) and topological compact solitary waves.

In this paper, we will deal with compact solitary waves—our interest being primarily in topological ones, hence the choice of a periodic on-site potential, see below—and use the word compacton as a synonym for compact solitary wave.

The possible relevance of compactons for applications as well as for theoretical developments need not be emphasized; here we only quote some papers dealing with their theory and applications [9, 10, 12, 13, 15–17, 25–28, 30, 34, 35]. We also mention that certain theories exhibit an intriguing duality between solitons and compactons [21].

Recently, Saccomandi and Sgura [32] have shown—in the context of a (Peyrard–Bishop type [7, 23]) dynamical model of the DNA macromolecule—that compact solitary waves can arise in a uniform nonlinear chain with smooth on-site potential, i.e. in an infinite uniform chain with variables ξ_n at site n ($n \in \mathbf{Z}$) interacting via a nearest neighbour interaction potential $V_s(\xi_n, \xi_{n+1})$ and subject to an on-site periodic potential $V_p(\xi_n)$.[4]

In particular, they considered the case where the nearest neighbour interaction is described by an anharmonic potential: with $\Delta_n = (\xi_{n+1} - \xi_n)$, choose, e.g., $V_s(\Delta) = K_1\Delta^2/2 + K_2\Delta^4/4$. In this case, the model supports compact solitary waves travelling with a speed determined by the elastic constant K_1 and the coefficient m playing the role of mass in the kinetic energy term.

In this paper, we consider the case of a linearly elastic chain, i.e. a harmonic first neighbours interaction $V_s(\xi_{n+1}, \xi_n) = (K/2)(\xi_{n+1} - \xi_n)^2$, and a possibly non-smooth on-site periodic potential (this will lead to topological compactons).

We will actually work with the continuum approximation for the chain and substitute the infinite array of coupled ODEs by a nonlinear wave equation. We stress that albeit our physical motivation for this study originated in chain models—which is the reason to start our discussion with these—our findings apply *a fortiori* to the class of nonlinear wave equations (8), see below.

We will find that for suitable properties of the on-site potential near its minima, the chain also supports travelling compact solitary waves, see section 6. In this case, there is moreover some wider freedom in the speed of the solitary waves.

The mechanism at the basis of this phenomenon is quite simple, and hence we expect it to be widely applicable physically; on the mathematical side, its understanding stems from the discussion of compact solitary waves given by Saccomandi in [31] and amounts to non-uniqueness of solutions in a certain auxiliary equation, see section 5.

Once the basic mechanism coupling non-smoothness of the on-site potential at its minima and compact solitary waves is understood, it can be generalized in different ways, as discussed in section 7 and shown by example in section 8, to frameworks more general than the simple one considered initially.

We can also consider and describe, with no substantial extra effort, *multi-compactons*; from the mathematical point of view, these display an intriguing phenomenon of non-uniqueness in the reduction from the second-order Newton equation to the first-order equation expressing conservation of energy. See appendix A for a discussion.

[4] With, e.g., $T_n = m\dot{\xi}_n^2/2$, the kinetic term for the variable ξ_n, the Saccomandi–Sgura model is described by the Lagrangian $\mathcal{L} = \sum_n \{(m/2)\dot{\xi}_n^2 - [V_p(\xi_n) + V_s(\xi_{n+1} - \xi_n)]\}$. The Euler–Lagrange equations are then of course $\ddot{\xi}_n = -(1/m)[(\partial V_p(\xi_n)/\partial \xi_n) + (\partial V_s(\xi_n - \xi_{n-1})/\partial \xi_n) + (\partial V_s(\xi_{n+1} - \xi_n)/\partial \xi_n)]$.

The physical interest of compactons lies in that they are solitary waves with strictly compact support; this also means that the energy is strictly localized in these field configurations. It is natural to ask if, for equations supporting compactons, there are solutions which are not travelling waves but still display strict localization of energy. We present our conjectures and heuristic arguments about this question in appendix B.

2. The chain model

We consider a uniform chain with sites indexed by an integer $n \in \mathbf{Z}$ and variables $\xi_n(t)$ at each site. The evolution of these is governed by a Lagrangian $\mathcal{L} = T - U$ where the kinetic energy is

$$T = \sum_n \frac{m}{2} \dot{\xi}_n^2 \tag{1}$$

and the potential energy $U = \widehat{U}_p + \widehat{U}_s$ is the sum of an on-site potential

$$\widehat{U}_p = \sum_n V(\xi_n) \tag{2}$$

and a first neighbours interaction potential depending only on the difference between the values of variables associated with first neighbouring sites,

$$\widehat{U}_s(\xi_n, \xi_{n+1}) = \sum_n V_s(\xi_{n+1} - \xi_n). \tag{3}$$

More specifically, we will consider the case of a linearly elastic chain, i.e. the V_s potential will be assumed to be harmonic,

$$V_s(\xi_{n+1} - \xi_n) = (k_s/2)(\xi_{n+1} - \xi_n)^2. \tag{4}$$

As for the on-site potential $V(x)$, we will assume it to be periodic and continuous, but not necessarily differentiable[5].

For ease of notation we will fix the period to be 2π and the absolute minimum to be reached for $x = 2k\pi$ (this can be achieved by a change in the origin and scale of x); again for ease of notation we also fix, by adding a suitable inessential constant to V, $V(0) = 0$. We also assume for ease of discussion that there are no other points $x \neq 2k\pi$ with $V(x) = 0$ (that is, the minima are non-degenerate in $[0, 2\pi]$).

Thus, our general Lagrangian will be

$$\mathcal{L} = \sum_n (m/2)\dot{\xi}_n^2 - (K/2)(\xi_{n+1} - \xi_n)^2 - V(\xi_n), \tag{5}$$

with the corresponding Euler–Lagrange equations

$$m\ddot{\xi}_n = k_s(\xi_{n+1} - 2\xi_n + \xi_{n+1}) - \frac{\partial V(\xi_n)}{\partial \xi_n}. \tag{6}$$

[5] We can think of ξ_n as describing the displacement of the particle at site n orthogonally to the chain direction; no confusion should be made between the lattice period δ (see next section) and the on-site potential period, chosen below to be 2π.

3. The continuum limit

If now we pass to the continuum limit, i.e. consider solutions which vary slowly in space compared with the inter-site length scale δ, the array of variables $\{\xi_n(t)\}$ is replaced by a field $\Phi(x,t)$, where $x \in \mathbf{R}$ is the coordinate along the chain and $\Phi(n\delta,t) \approx \xi_n(t)$. In this limit, equation (6) becomes

$$m\Phi_{tt} = k_s\delta^2\Phi_{xx} - \frac{\partial V(\Phi)}{\partial\Phi} + O(\delta^4). \tag{7}$$

We let then δ tend to zero, changing at the same time k_s so that $k_s\delta^2 = K$ remains constant and get

$$m\Phi_{tt} = K\Phi_{xx} - \frac{\partial V(\Phi)}{\partial\Phi}. \tag{8}$$

We thus arrived at a wave equation; if $V(\Phi)$ is not harmonic, it will be a *nonlinear wave equation*.

The same equation would be obtained by considering the continuum limit directly on the model, i.e. on the Lagrangian (5); this amounts to considering the field Lagrangian

$$\mathcal{L} = \int (m/2)\Phi_t^2 - (K/2)\Phi_x^2 - V(\Phi)\,\mathrm{d}x. \tag{9}$$

This yields again (8) as the associated Euler–Lagrange equation.

Note that we should specify a function space to which $\Phi(x,t)$ is required to belong. The natural physical condition is that of finite energy; in terms of boundary asymptotic conditions, we require that the state of the system corresponds to a stable equilibrium (in field theoretic language, a vacuum) at $x \to \pm\infty$. Thus, the boundary conditions complementing (8) will be

$$\lim_{x\to\pm\infty}\Phi_t = 0, \qquad \lim_{x\to\pm\infty}\Phi_x = 0, \qquad \lim_{x\to\pm\infty}\Phi = \phi_\pm. \tag{10}$$

Here $\phi_\pm$ are minima for V; with our assumption above for V this means $\phi_\pm = 2k_\pm\pi$. Actually we can always choose the origin of the x scale such that $k_- = 0$, and hence we will in the end require that

$$\lim_{x\to-\infty}\Phi(x,t) = 0, \qquad \lim_{x\to+\infty}\Phi(x,t) = 2n\pi. \tag{11}$$

Important remark. Albeit our original physical motivation originated in chain models generalizing the simple pendulum chain (which leads to sine-Gordon equation) [4], we will from now on work at the continuum level. Thus, one could just start from the nonlinear wave equation (8), or equivalently from the field Lagrangian (5), with the limit conditions (11).

4. Travelling wave solutions

Travelling wave solutions (with speed v) are solutions of the form

$$\Phi(x,t) = \varphi(x - vt) := \varphi(z). \tag{12}$$

By plugging this *ansatz* into (8) we obtain the equation

$$mv^2\varphi'' = K\varphi'' - \frac{\partial V(\varphi)}{\partial\varphi}. \tag{13}$$

It is convenient to rewrite this as

$$\varphi'' = -\frac{\partial W(\varphi)}{\partial\varphi}, \tag{14}$$

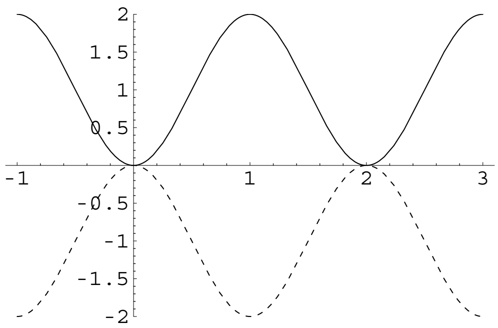

Figure 1. Physical versus effective potential for $\mu < 0$ (in this case $\mu = -1$). For negative μ the minima of the physical on-site potential (solid curve) are maxima of the effective potential (dashed curve) and there are motions approaching different minima of V (hence maxima of W) for $z \to \pm\infty$.

where we have defined the *effective potential*

$$W(\varphi) = \frac{1}{mv^2 - K} V(\varphi) := \frac{1}{\mu} V(\varphi). \tag{15}$$

Equation (14) describes the motion (in the 'time' z) of a particle of unit mass in the potential W; the latter is the same as V apart from the multiplicative constant μ^{-1}, which could be of any sign. In particular,

$$\mu < 0 \Leftrightarrow |v| < v_* = \sqrt{\frac{K}{m}}. \tag{16}$$

The limit conditions (10) are inherited by (14). It follows from them that for $z \to \pm\infty$ the value of φ' should go to zero, and φ should approach the constant value $2k_\pm\pi$, i.e. a minimum of $V(\varphi)$. (As in (11) we can change the origin of φ and actually require $\lim_{z\to-\infty} \varphi = 0$, $\lim_{z\to+\infty} \varphi = 2n\pi$.)

In the description in terms of an effective potential, as we assumed $V(0) = 0$, they require that the 'effective energy' $E = (1/2)(\varphi')^2 + W(\varphi)$ is set to zero,

$$E = \frac{(\varphi')^2}{2} + W(\varphi) = 0. \tag{17}$$

For $\mu > 0$ this is incompatible with nontrivial motions, i.e. the limit conditions uniquely select the solution $\varphi(z) \equiv 0$.

On the other hand, for $\mu < 0$ minima of V are maxima of the effective potential W (see figure 1) and it is possible to have nontrivial motions joining two adjacent minima of the potential V.

In the following, we will focus on the solutions with

$$\lim_{z\to-\infty} \varphi = 0, \qquad \lim_{z\to+\infty} \varphi = 2\pi \tag{18}$$

for ease of discussion. We will refer to these as *kinks*.

Note that (17) allows us to deal with a first-order equation in between inversion points, i.e., yields

$$\varphi' = \pm\sqrt{-2W(\varphi)}; \tag{19}$$

the boundary conditions (18) require that the positive determination of the square root is selected. (The non-uniqueness of the first-order ODE associated with energy conservation will be discussed in more detail in appendix A.)

5. Non-uniqueness of solutions

From a mathematical point of view, at the heart of the problem is a one-dimensional autonomous first-order equation for $y(x)$,

$$y' = f(y), \tag{20}$$

with f continuous but not necessarily Lipschitz.

By Peano's theorem every initial value problem has a solution. Uniqueness and non-uniqueness can be discussed directly, using the separability of the equation. The text by Walter [36] contains a careful analysis of this scenario; from this basic reference we will draw our conclusions. (Differential equations with non-Lipschitz nonlinearities are also discussed in [1, 22].)

We will discuss the following setting: f is defined on some real interval $[y_-, y_+]$ with $y_- < y_+$; y_- and y_+ are stationary, and there is no further stationary point between these two. We investigate one case extensively:

Let $y_- < y_0 < y_+$ and assume $f(y_0) > 0$. Then

(i) The initial value problem (20) with $y(x_0) = y_0$ has a unique local solution ψ, which is increasing.

(ii) If the integral

$$\int_{y_0}^{y_+} \frac{\mathrm{d}y}{f(y)} \tag{21}$$

diverges then ψ can be extended to $[x_0, +\infty)$, with limit y_+, which is not attained. If the integral (21) converges then there is a smallest x^* such that ψ can be extended to $[x_0, x^*]$, $\psi(x^*) = y_+$ and $\psi(x) < y_+$ for $x < x^*$. Moreover, this solution can be continued to any interval (x^*, b) by the constant y_+.

(iii) If the integral

$$\int_{y_-}^{y_0} \frac{\mathrm{d}y}{f(y)} \tag{22}$$

diverges then ψ can be extended to $(-\infty, x_0]$, with limit y_-, which is not attained. If the integral (22) converges then there is a largest x_* such that ψ can be extended to $[x_*, x_0]$, $\psi(x_*) = y_-$ and $\psi(x) > y_-$ for $x > x_*$. Moreover, this solution can be continued to any interval (a, x_*) by the constant y_-.

(iv) Assume that both integrals (21), (22) converge. Then every solution ϕ that connects y_- and y^+ (thus $\phi(x) = y_-$ for $x \to -\infty$ and $\phi(x) = y_+$ for $x \to \infty$) is a translate of the extended solution ψ.

The proof of (i)–(iii) follows directly from [36], while part (iv) is due to the autonomy of (20). Thus we see that there is, essentially, just one kink connecting the two stationary solutions y_- and y_+ (it is relevant we assumed there is no stationary point in between these two) and that the 'energy integral'

$$\int_{-\infty}^{+\infty} [\phi'(y)]^2 \mathrm{d}y \tag{23}$$

is the same for all such solutions.

We also recall that the integral (21) converges whenever $|f(y)| \approx (y_+ - y)^s$ for some s, $0 < s < 1$, and similarly for (22). Thus we have quite simple criteria in many examples.

These results also hold, *mutatis mutandis*, in the case $f(y_0) < 0$.

6. Compact travelling waves and regularity

We now return to periodic potentials. The *width* L of the kink will be defined as the length of the z interval in which $\varphi' \neq 0$, i.e. the constant width of the spatial region (at any time t) in which the field is not *exactly* in a minimum of V.

In order to evaluate this, we can follow a standard procedure in mechanics (see, e.g., [18]). Conservation of effective energy E implies that

$$\frac{\mathrm{d}\varphi}{\mathrm{d}z} = \pm\sqrt{2[E - W(\varphi)]}; \tag{24}$$

the boundary conditions (18) lead us to assume the positive sign here. Hence we can express L as $L = \int_0^{2\pi} [2[E - W(\varphi)]]^{(1/2)}\,\mathrm{d}\varphi$. Recalling (17) and $W = \mu^{-1}V$ with $\mu < 0$, and writing $\alpha = \sqrt{|\mu|/2}$ for ease of notation, we get

$$L = \alpha \int_0^{2\pi} \frac{1}{\sqrt{V(\varphi)}}\,\mathrm{d}\varphi. \tag{25}$$

Recalling the arguments from the previous section, we have that *If* $V(\varphi) \approx \varphi^{s_\pm}$ *for* $\varphi \to 0^\pm$, *and* $0 \leqslant s_\pm < 1$, *then the kink solutions have finite width*, $L < \infty$.

In the following examples we will take $s_+ = s_- = s$ for ease of notation and discussion.

The presence of travelling waves with finite width can also be understood looking back at (24). Using again $E = 0$, $W = \mu^{-1}V$ and $\mu < 0$, this reads $\mathrm{d}\varphi/\mathrm{d}z = \alpha\sqrt{V(\varphi)}$. For $\varphi \approx 0$ and with our assumptions on the behaviour in this region, we may consider the limiting case

$$\frac{\mathrm{d}\varphi}{\mathrm{d}z} = \alpha\varphi^s. \tag{26}$$

When $0 < s < 1$, several solutions can merge at this point. In particular, (26) admits the trivial solution $\varphi_0(z) = 0$ together with the family of nontrivial solutions

$$\varphi_1(z) = [(1-s)\alpha(z - z_0)]^{1/(1-s)}. \tag{27}$$

For $z \to z_0$, this merges with φ_0. A similar analysis applies near $\varphi = 2\pi$.

Our assumptions on V guarantee that the ODE (24) has no singular point for $\varphi \neq 2k\pi$. Thus, a solution $\varphi_*(z)$ to the full equation (24), behaving like (27) for $\varphi \to 0^+$, is unique and can be (uniquely) prolonged until φ comes to 2π; for $\varphi \to (2\pi)^-$ there is an analogous description.

Thus we have different kinds of solutions: trivial ones, with $\varphi(z) = 2k\pi$, and solutions joining adjacent minima of V. As these merge in $\phi = 2k\pi$, we can construct new solutions by passing from one branch to another one (merging with the first) at these point.

At the merging points these new solutions will be differentiable by construction, but will not be twice differentiable if $1/2 < s < 1$.

In particular, we can consider a solution of the kind

$$\varphi(z) = \begin{cases} 0 & \text{for } z < z_0 \\ \varphi_*(z) & \text{for } z_0 < z < z_1 \\ 2\pi & \text{for } z > z_1. \end{cases} \tag{28}$$

This corresponds indeed to the kink solution considered above and $(z_1 - z_0) = L$. Note that the limit conditions (11) force the choice for large $|z|$, which must correspond to an equilibrium (a *vacuum* in field theory language).

Note that in principles one could have solutions going back and forth between 0 and 2π several times before resting on one of these equilibria[6]; however, these will have higher energy than the one with only one transition, which should hence be selected for further discussion.

Similarly, one could combine two (or more) kink solutions to obtain a multi-kink, which is a special type of *multi-compacton*, see below for a discussion.

Going back to physical energy for the compact wave field configuration, we note that the energy contribution of the trivial parts of the solution is strictly zero (in these regions the field is in a vacuum) and the contribution of each transition is a constant determined by the equation. The energy of the field is contained in a moving region of width L.

Finally, we recall that when starting from a chain model, we have to ask that the solution varies little over the inter-site distance δ; this implies in particular we should have $\delta \ll L$ for the continuum approximation to make sense (and the compacton solution to apply) in this context.

7. Discussion

It maybe appropriate to stress some points concerning comparison of the compacton solutions in our model of linear elastic chain on the one hand with the more familiar soliton ones[7] and on the other hand with the case of nonlinear elastic chains.

(1) This construction makes full use of the connection between compact waves and lack of regularity emphasized in [31].

(2) The speed v of the travelling compact wave is—in this simple model—a free parameter. This is like the case of solitons, and at difference with what was observed for nonlinear elastic chains with regular on-site potentials [32], where the speed of compactons is uniquely fixed by (the values of parameters appearing in) the equation.

(3) There is a well-defined relation between the speed and the width and energy of the compact kink solutions. This is again similar to what happens for solitons [5, 7, 11, 24].

(4) It is easy to conclude from section 5 that one could consider compacton solutions travelling between two non-adjacent minima of $V(\varphi)$ (e.g. with $\varphi(-\infty) = 0$ and $\varphi(+\infty) = 4\pi$) combining two or more 'elementary kink' solutions, possibly separated by a 'pausing' region where the field is in a vacuum. These multi-kink solutions will have more than one excited region; the 'elementary components' will not interact among themselves as their supports do not intersect. (See also appendix A for more details related to this remark.)

(5) The sum of two compacton solutions having disjoint support will of course be still a solution to the full equation. Note that for equations supporting compact waves with arbitrary speed, different elementary components initially separated could come to interact in the course of their evolution.

(6) We discussed the case of a periodic potential $V(\xi)$, but this is not necessary for the mechanism described here to work. What really matters is that V is sub-quadratic in the relevant minima (those to which the solution is asymptotic for large $|z|$ and hence for $x \to \pm\infty$).

[6] Calling the solution joining 2π to 0 an *anti-kink*, these solutions would be a combination of kinks and anti-kinks. Kinks carry a topological charge $c_k = +1$, while anti-kinks carry a topological charge $c_{ak} = -1$; thus kink/anti-kink pairs can always be inserted with no harm for the global topological charge and hence for the boundary conditions (18), but they are energetically costly.

[7] The main difference is of course in the compact support of solutions. Solitons are uniformly analytic (extend to a holomorphic function in strips of the type $|\mathrm{Im}(z)| < T$) and decay exponentially to zero (see, e.g., [3, 5, 14, 20]). The analyticity results automatically excludes the existence of interval where the travelling wave is identically zero.

(7) Thus, compactons are still possible if V is not periodic but has (at least) two degenerate local minima $\xi_\pm$ (with no points in between with $V(\xi) \leqslant V(\xi_\pm)$) and that it is sub-quadratic in these; this will be shown in example 2.

(8) It is not difficult to see that one could as well have semi-compact travelling waves. This would happen, e.g., if one has a degenerate minimum, $V(\xi_-) = V(\xi_+) = 0$ (with no points in between with $V(\xi) \leqslant V(\xi_\pm)$), but with different regularity properties in ξ_+ and ξ_-, e.g. $V(x) \approx (x - \xi_+)^{2s}$ with $1/2 < s < 1$ near ξ_+ and $V(x) \approx (x - \xi_-)^2$ near ξ_-. In this case there will be a travelling wave which is exactly zero ahead of $x_c(t) = vt$, but has semi-infinite support $S_t = \{x < x_c(t)\}$.

(9) Similarly, one could also have compact travelling bumps (rather than kinks), based on the same mechanism: due to the ambiguity of the sign in equation (24), we may combine solutions going 'up' and 'down'. For this we need that there is a single local minimum in ξ_0 and V is sub-quadratic in it, with another point ξ_1 such that $V(\xi_1) = V(\xi_0)$, $V'(\xi_1) \neq 0$, and no other points ξ in between ξ_0 and ξ_1 where $V(\xi) = V(\xi_0)$; this will be shown in example 4.

(10) For reaction–diffusion systems (appearing, e.g., in models for chemical reactions), it may happen that the concentration $u(x, t)$ becomes a 'dead core' type solution; that is, the set where $u = 0$ is an open unbounded domain. Dead core solutions appear for steady-state diffusion–reaction equations having monotone kinetics. It turns out that it is necessary to consider non Lipschitz nonlinearities and by the comparison principle one is led to study ODEs of the type $u' = -u^s$ with $0 < s < 1$, hence quite similar to those considered here. More generally, for some model reaction–diffusion equations a region of zero reaction concentration may be formed in finite time [2, 33]. This seems to point out that compactons (in this case, more precisely, semi-compact travelling waves) can emerge naturally from the dynamics as asymptotic solutions for large t, with rather general initial data.

(11) As pointed out in section 2, the class of models discussed here can be thought as describing (in the continuum limit) transversal motions of particles on a uniform chain (one-dimensional lattice). One could also think of considering motions of the particles in the lattice direction (this case would introduce a relation between the potential period and the lattice one); one would then have Davydov-like solitons [8] with compact support. This case is of obvious physical interest and worth further investigation.

8. Examples

In this section, we discuss some examples where the relation between non-smoothness of the on-site potential at its minima and the appearance of compact waves is apparent. We chose simple potentials in order that our examples are actually able to clarify the issue.

8.1. Example 1: kink with a periodic on-site potential

We will first consider the simple example

$$V(\Phi) = |\sin(\Phi/2)|. \tag{29}$$

In this case $s = 1/2$; equation (24) reads

$$\frac{d\varphi}{dz} = \alpha\sqrt{|\sin[\varphi(z)/2]|}. \tag{30}$$

This can be solved in terms of elliptic functions; more precisely, we have

$$\varphi(z) = \pi - 4\mathcal{A}[-(\alpha/4)(z - z_0), 2], \tag{31}$$

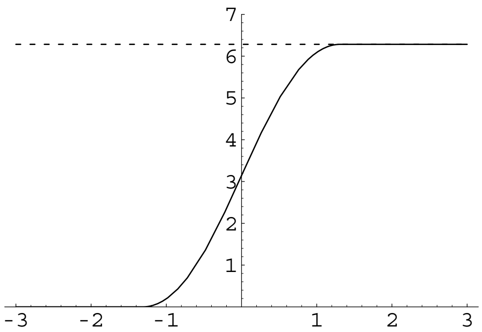

Figure 2. The (compact wave) kink solution joining 0 and 2π for the potential $V(\varphi) = |\sin(\varphi/2)|$ (example 1). Here we use $\beta = 1$, i.e. $\mu = -1/8$.

where $\mathcal{A}$ is the Jacobi amplitude function (this is the inverse of the elliptic integral of first type F: if $u = F[x, k]$, then $x = \mathcal{A}[u, k]$) and z_0 is an integration constant. The solution (31) is plotted in figure 2.

We recall that $\mathcal{A}[0, k] = 0$, hence z_0 represents the point at which $\varphi(z_0) = \pi$ (i.e. it is half-way between the adjacent maxima of the effective potential). We will set $z_0 = 0$, and in view of the symmetry of the potential (29) this will give a symmetric kink solution, $\varphi(z) - \pi = \pi - \varphi(-z)$.

With the choice $z_0 = 0$, and writing $\beta := (\alpha/4)^{-1} := \sqrt{8|\mu|}$, our solution becomes

$$\varphi_*(z) = \pi - 4\mathcal{A}[-z/\beta, 2]. \tag{32}$$

This takes the critical values 0 and 2π, i.e. solves $\varphi_*(z_+) = 2\pi$ and $\varphi_*(z_-) = 0$, for $z_\pm = \pm\beta F[\pi/4, 2]$; thus the width of the kink is $L = 2\beta F[\pi/4, 2] = (4\sqrt{2|\mu|})F[\pi/4, 2]$.

Needless to say, the kink solution will be given by

$$\varphi(z) = \begin{cases} 0 & \text{for } z < z_- \\ \varphi_*(z) & \text{for } z_- < z < z_+ \\ 2\pi & \text{for } z > z_+. \end{cases} \tag{33}$$

In the case $\beta = 1$ (different cases can be reduced to this by a rescaling in z), the roots, i.e. the points at which the solution φ_* given in (32) meets the solutions $\varphi = 0$ and $\varphi = 2\pi$, are given by $z_\pm \simeq \pm 1.311$.

If we consider the first and second derivatives of φ_*, we have (here dn, sn, cn are Jacobi elliptic functions) $\varphi_*'(z) = 4\mathtt{dn}[-z, 2]$, $\varphi_*''(z) = 8\mathtt{cn}[-z, 2]\mathtt{sn}[-z, 2]$; evaluating these at $z = z_\pm$, it results $\varphi_*'(z_\pm) = 0$, $\varphi_*''(z_\pm) = \mp 4$. This shows that the kink solution (33) is $\mathcal{C}^1$ but not $\mathcal{C}^2$.

Finally, we note that it is obviously easy to build multi-kink solution, as mentioned in the discussion above (see section 7).

8.2. Example 2: kink with a bistable on-site potential

Let us consider an elastic chain with an on-site bistable potential, (see figure 3),

$$V(\Phi) = (|\Phi^2 - 1|)^{2p} \qquad (0 < p < 1) \tag{34}$$

(for $p \geqslant 1$ we would get a travelling wave solution with non-compact support). In the continuum approximation the limit conditions enforce

$$\lim_{x\to\pm\infty} \Phi^2(x, t) = 1. \tag{35}$$

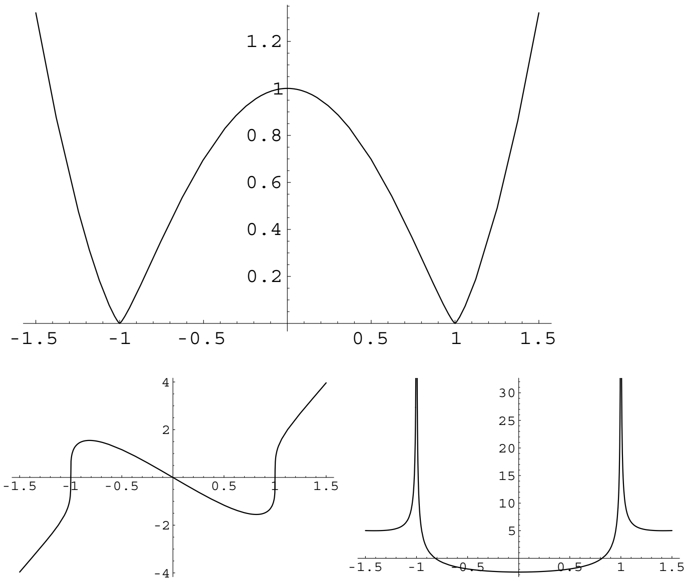

Figure 3. Above: the potential $V(\Phi)$ considered in example 2 for $p = 5/8$. Below: the first (left) and second (right) derivatives for the same potential. Note $V(\Phi)$ is $\mathcal{C}^1$ but not $\mathcal{C}^2$ near the minima at $\Phi = \pm 1$.

The Euler–Lagrange equation for travelling waves turns out to be

$$(mv^2 - K)\varphi'' = -(\partial V(\varphi)/\partial\varphi), \tag{36}$$

and this is seen as the equation describing the motion of a particle of unit mass in the effective potential $W(\varphi) = \mu^{-1}V(\varphi)$, where as usual we defined $\mu = (mv^2 - K)$. The boundary conditions ensure that nontrivial motions are possible only for $\mu < 0$ (which we assume from now on), i.e. for $|v| < \sqrt{K/m}$; again due to the boundary conditions, we should require that the conserved total energy has the same value as in the minima of V, i.e. zero,

$$E := (1/2)(\varphi')^2 + W(\varphi) = 0.$$

Thus, by conservation of energy,

$$\frac{\mathrm{d}\varphi}{\mathrm{d}z} = \sqrt{-2W(\varphi)} = \sqrt{2/|\mu|}|\varphi^2 - 1|^p. \tag{37}$$

This has trivial solutions $\varphi_\pm(z) = \pm 1$, together with nontrivial solutions joining them. In particular, the solution $\varphi_*(z)$ travelling from $\varphi_- = -1$ to $\varphi_+ = +1$ is obtained via a simple integration by parts,

$$\frac{\mathrm{d}\varphi_*}{(\varphi_*^2 - 1)^p} = \alpha\,\mathrm{d}z; \tag{38}$$

here we wrote $\alpha = \sqrt{2/|\mu|}$ and used $|\varphi| \leqslant 1$ to eliminate the absolute value. Setting the integration constant so that $\varphi_*(0) = 0$, this is solved (in implicit form) in terms of hypergeometric functions and yields

$$\varphi_*\,{}_2F_1[1/2, p, 3/2, \varphi_*^2] = z. \tag{39}$$

The function $\varphi_*(z)$ obtained by inverting this is plotted in figure 4.

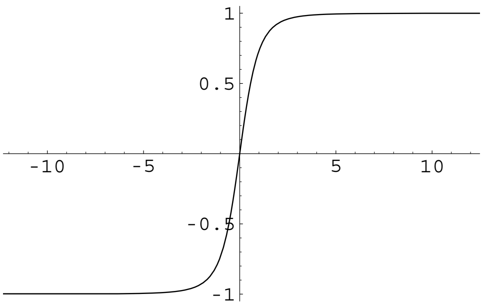

Figure 4. The kink solution for example 2. Here we used (as in figure 3) $p = 5/8$.

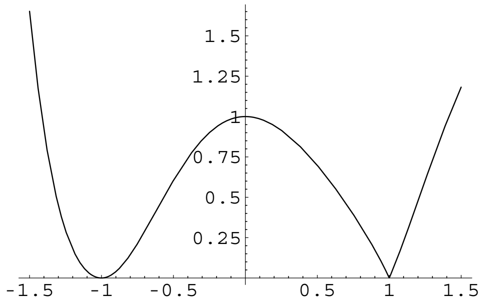

Figure 5. The potential for example 3.

As for the roots of $\varphi_*(z_\pm) = \pm 1$, they are $z_\pm = \pm[\sqrt{\pi}\Gamma(1-p)]/[2\Gamma(3/2-p)]$, and hence the width of the kink solution

$$\widehat{\varphi}(z) = \begin{cases} -1 & \text{for } z < z_- \\ \varphi_*(z) & \text{for } z_- < z < z_+ \\ 1 & \text{for } z > z_+ \end{cases} \tag{40}$$

turns out to be $L = (z_+ - z_-) = [\Gamma(1-p)/(\Gamma(3/2-p))]\sqrt{\pi}$. Note this diverges for $p \to 1$.

8.3. *Example 3: semi-compact kinks*

As mentioned above, the same mechanism leading to compact waves can lead to semi-compact ones. An example is provided by a linear elastic chain with on-site potential

$$V(x) = (|x^2 - 1|)^{2p(x)} \tag{41}$$

with a suitable function $p(x)$; here suitable means that $p(-1) = 1$ while $1/2 \leqslant p(1) < 1$. In order to deal with a concrete example, we choose (see figure 5)

$$p(x) = (3 - x)/4. \tag{42}$$

Proceeding as usual, and writing $\mu = (mv^2 - K)$, one gets to describing travelling waves as the motion of a particle of unit mass in the effective potential $W(\varphi) = \mu^{-1}V(\varphi)$. We

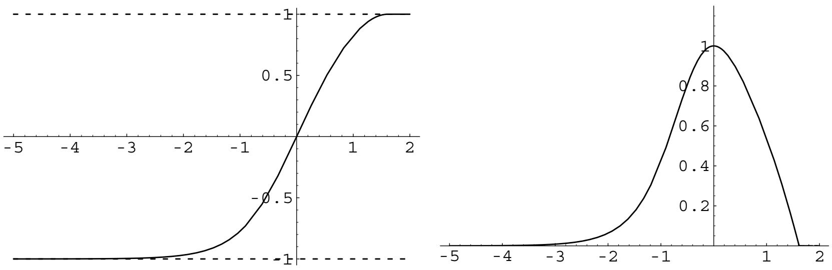

Figure 6. Example 3. Numerical integration of (43) for the choice (42); initial conditions chosen as $\varphi(0) = 0$. Left: the function $\varphi(z)$ (solid curve) together with the trivial solutions $\varphi_{\pm}(z) = \pm 1$ (dashed lines). Right: the first derivative $\varphi'(z)$. Note that $\varphi'(z)$ is discontinuous when $\varphi(z)$ meets the critical point $\varphi = 1$.

assume $\mu < 0$; now $\varphi = \pm 1$ are maxima for W, and for $\varphi \in [-1, 1]$ the dynamics is governed by

$$\frac{d\varphi}{dz} = \alpha\sqrt{V(\varphi)} = \alpha(1-\varphi^2)^{p(\varphi)} \tag{43}$$

with $\alpha = \sqrt{2/|\mu|}$.

We were not able to solve this in closed form; a numerical integration is however possible. See figure 6 for the case (42); note that the obtained kink solution is $\mathcal{C}^1$ (but not $\mathcal{C}^2$).

8.4. Example 4: compact travelling bump waves

Here we consider a potential supporting a travelling compact bump (rather than kink) solution. By this, we mean a solution $\Phi(x, t)$ which is exactly zero (or more generally corresponds to a single vacuum) out of a travelling region. Note in this case Φ goes to zero at infinity in both the positive and negative directions.

For this to happen, the effective potential $W(\varphi)$ must support a homoclinic trajectory doubly asymptotic to a maximum ϕ_0 of W, hence a minimum for V, at which $V(\varphi)$ is subquadratic.

An example of this situation is provided by

$$V(x) = (1+x)(|x-1|)^{(2p(x))}, \tag{44}$$

where $1/2 \leqslant p(1) < 1$; we can choose, e.g., $p(x) = (1/4)[3 + (\tanh(x-1))^2]$ or similarly

$$p(x) = 1 - (1/2)\exp[-(x-1)^2]; \tag{45}$$

thus $p(x) \simeq 1$ except in a small region around $x = 1$. The potential obtained with the prescriptions (44), (45) is shown in figure 7.

The bump solution $\varphi(z)$ satisfies $\varphi(z) \in [-1, 1]$ for all z. If we fix the arbitrary integration constant so to satisfy

$$\varphi(0) = -1 \tag{46}$$

the bump solution is described by

$$\frac{d\varphi}{dz} = \pm\sqrt{V[\varphi(z)]}, \tag{47}$$

with the sign chosen appropriately for each piece. Once again we are not able to solve this explicitly, but the equation can be solved numerically; the result of the numerical integration is shown in figure 8. Note again that the obtained kink solution is $\mathcal{C}^1$ but not $\mathcal{C}^2$ in view of the requirement $1/2 < p(1) < 1$.

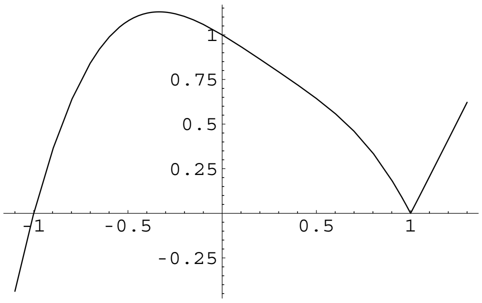

Figure 7. The potential for example 4. This is the potential $V(x)$ given by (44) with the prescription (45).

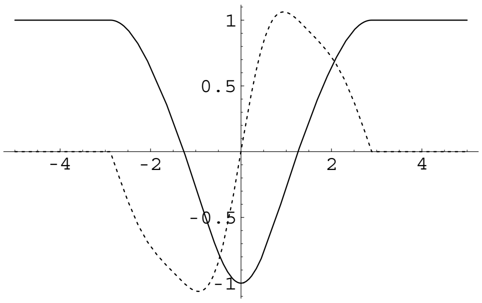

Figure 8. Example 4. Numerical solution for the bump solution $\varphi_*(z)$, obtained solving (47) and imposing (46). We show $\varphi_*(z)$ (dotted curve) together with $\varphi_*'(z)$ (solid curve).

9. Conclusions

We have considered linearly elastic chains with a periodic on-site potential $V(\xi)$ which is continuous but non-smooth at its minima. We considered configurations varying slowly from one site of the chain to the next one, and passed to consider a continuum version of our problem, formulated in terms of a field $\Phi(x, t)$.

Our model is then described by the nonlinear wave equation (8) with the limit conditions (11) or equivalently by the field Lagrangian (5) with the finite energy condition. We focused on travelling wave solutions, which led to a formulation in terms of an effective potential $W(\varphi) = \mu^{-1}V(\phi)$, where μ is a negative parameter related to the characteristics of the chain as well as to the speed v of the travelling wave; the latter must be smaller than a limiting speed, see (16).

We have shown that the non-smoothness of V at its minima lead to the appearance of kink travelling waves with compact support. The same mechanism—discussed in sections 5 and 6—can also be at play when other kinds of compact (or semi-compact) travelling waves appear, see the discussion in section 7.

This seems to be a new mechanism for the generation of compact travelling waves, as previous works in the literature only considered smooth on-site potentials.

In particular, we stress that in this way one can have compactons in an elastic chain even when the elastic interaction is completely linear: thus the present mechanism, based on a sub-quadratic behaviour of the potential at its minima, complements the one devised by Saccomandi and collaborators [9, 31, 32] in which the compactons arise from a balance between (effective) nonlinear elasticity in the chain and (effective) super-quadratic behaviour of the potential at its minima; the 'effective' here refers to the fact it suffices this behaviour describes relevant terms in the equations—or Lagrangian—after the travelling-wave reduction.

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Appendix A. Multi-compactons

Consider a continuous T periodic potential $V(x)$ satisfying $V(x) \leqslant E_0$ on all of $x \in \mathbf{R}$, with more specifically $V(kT) = E_0$ for $k \in \mathbf{Z}$, and $V(x) < E_0$ for $x \in \mathbf{R}\backslash\mathbf{Z}$.

As a model example we can take the 2π periodic potential (with $s > 0$)

$$V(x) = E_0 - V_0(x) = E_0(1 - |\sin(x/2)|^s). \tag{A.1}$$

We note that this $V(x)$ is $\mathcal{C}^2(\mathbf{R})$ if and only if $s \geqslant 2$; it is $\mathcal{C}^1(\mathbf{R})$ if and only if $s > 1$; it is Hölder $\mathcal{C}^s(\mathbf{R})$ if $s \notin \mathbf{N}$. Moreover, setting $f(x) := \sqrt{V_0(x)}$, f is Lipschitz if and only if $s \geqslant 2$. We also have that $\int_a^b [f(y)]^{-1} \mathrm{d}y < +\infty$ for all a, b if and only if $s < 2$.

Consider now trajectories of particles on the energy level E_0: these are given by $\dot{x}^2 + 2V(x) = 2E_0$, which is equivalent to

$$\dot{x}^2 = 2V_0(x). \tag{A.2}$$

There are equilibrium solutions $x \equiv 2k\pi$, $k \in \mathbf{Z}$. In the Lipschitz case the uniqueness holds, while for the non-Lipschitz case $s < 2$ the solutions are not unique. In this case one can show the following.

Set, for $k \in \mathbf{Z}$,

$$f_k(x) = \begin{cases} f(x) & \text{if } x \in [2k\pi, (2k+2)\pi], \\ 0 & \text{if } x \notin [2k\pi, (2k+2)\pi]. \end{cases} \tag{A.3}$$

Then $x(t)$ is a $\mathcal{C}^1(\mathbf{R})$ solution of (A.2) if and only if

$$\dot{x} = \sum_k \sqrt{2}\alpha_k f_k(x) \tag{A.4}$$

where $\alpha_k = \pm 1$.

If $\alpha_k = 1$ (respectively, $\alpha_k = -1$) for all $k \in \mathbf{Z}$, then (A.4) becomes

$$\dot{x} = \sqrt{2V_0(x)} \tag{A.5}$$

(respectively, $\dot{x} = -\sqrt{2V_0(x)}$) and we recover the case of a periodic on-site potential. Note however that an arbitrary sequence of α_k is allowed in the statement; this will lead to an arbitrary sequence of kinks and anti-kinks, combining together to build a *multi-compacton*.

One can derive similar assertions for non-periodic potentials provided $V_0(x)$ has a sufficiently rich structure of degenerate zeros (e.g., more than two zeros).

Appendix B. Strong localization of energy for general solutions

It is natural to wonder if such strong localization of energy as the one observed in our note is unique to travelling wave solutions or if instead the boundary conditions and the features of the on-site potentials we considered cause strong localization of energy also for general solutions. Roughly speaking, one expects this to be the case (possibly under some additional conditions), on the basis of the following heuristic argument.

Consider a field configuration $\Psi(x,t)$ satisfying the limit conditions and, say, decaying exponentially to $\Psi = 0$ for $x \to +\infty$. Consider a small variation $(\delta\psi)(x,t)$ in the asymptotic region, whose effect is of making sharper the transition from finite values to zero. That is, when $|\Psi(x,t)|$ is sufficiently small and x sufficiently large, we have $|\Psi(x,t)+\delta\psi(x,t)| < |\Psi(x,t)|$. This will produce an increase in the energy coming from the terms involving Φ_t and Φ_x, but a decrease in the term involving $V[\Phi(x,t)]$. The increase will be of the order $\Psi_x(\delta\psi)_x + \Psi_t(\delta\psi)_t$, while the decrease will be of the order $\Psi^{(2p-1)}(\delta\psi)$.

If moreover we have $|\nabla\Psi| \approx k|\Psi|$ (we stress this is not following automatically from the exponential decay of Ψ; see [6] for a counterexample) the net result of the variation depends on the ratio $\nu := |\Psi^{(2p-1)}|/|\Psi| \approx |\Psi^{2(p-1)}|$ being greater or smaller than one for $\Psi \to 0$. For $p < 1$ this is greater than 1—and actually diverges in the limit—hence the net energy decreases under the variation $\delta\psi$, and field configurations with sharper decay to the limit behaviour will be favoured.

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